A NOTE ON LI'S PAPER

By

D. C. Rao

Indian Statistical Institute, Calcutta

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This note extends the main results in 'Fisher, Wright and path coefficients' of Li (1968) to any general mating system, from where Li's results can be obtained directly as particular cases. On these lines, using path coefficients, a general expression for the correlation between two half-sibs, with respect to a phenotypic measurement, has been given in this note.

The main purpose of this note is to extend some of the results obtained by Li (1968) under random mating, to any general mating system which involves an additional parameter. At the end of this note, a general expression for the correlation between two half-sibs with respect to a phenotypic measurement is given. The basic model considered here is the following :

Consider two alleles A and a at an autosomal locus, giving rise to three genotypes AA, Aa and aa. Let the two variables Z and Y denote the genic content and the measurement of a quantitative trait of the genotypes respectively. Thus, $Z=2, 1, 0$ (number of A-genes) for the genotypes AA, Aa and aa respectively, and let Y_2, Y_1 and Y_0 be the corresponding values of Y. Treating Z as the independent variable, the linear regression coefficient of Y on Z is denoted by b. The theoretical or expected values of Y from the above regression are denoted by L. The quantities σ_Z^2 and σ_L^2 denote the respective variances of Z and L, and $\gamma_{ZZ'}$ the correlation coefficient (for Z) between parent and offspring.

For the general mating system the genotypic frequencies are given by

$$AA : p^2+Fpq$$
$$Aa : 2pq(1-F)$$
$$aa : q^2+Fpq$$

such that

$$p+q=1 \quad \text{and} \quad 0 \leq F \leq 1$$

(F is known as the inbreeding coefficient). It may be remarked that each quantity derived here under the general mating system will be given an asterisk mark (*), at its base, and the plain symbols denoting the same quantities under random mating, e g. b^* denotes the linear regresssion coefficient of Y on Z under the general mating system whereas b denotes the same under random mating.

Under this set-up, the following observations are made :

(1) Average genic content of the population depends only on the gene frequency and is independent of the mating system, which is given by

$$E_*(Z)=2p \text{ for all } F,\ 0 \leq F \leq 1$$

(2) However, variation in the genic content depends on the mating system also, given by

$$\sigma_{Z_*}{}^2=2pq(1+F)$$

(3) Ignoring the environmental variation in Y (if there is environmental variation, then Y denotes the average over a given set of environments), the linear regression of Y on Z is sepecified by

$$b_*=b+\left(\frac{F}{1+F}\right)(Y_2-Y_0)$$

(4) The variance of the fitted values (L) is then given by

$$\sigma_{L_*}{}^2=2pq(1+F)b_*{}^2=\sigma^2_{Z_*}b_*{}^2$$

Parent-Offspring Correlation

For the general mating system, the author prefers to derive the joint distribution of the parent-child (one parent and one child) pairs rather clearly, since the mating frequencies are not popularly known. Distinguishing between both the parents, the nine genotypic matings, their frequencies and the corresponding segregation probabilities are shown in Table 1 [after Morton (1963)].

TABLE 1

General mating frequencies and the segregation probabilities

Mating		Segregation probabilities		
Type	*frequency*	*AA*	*Aa*	*aa*
(1)	(2)	(3)	(4)	(5)
$AA\times AA$	p^4+6p^3qF	1	0	0
$\times Aa$	$2p^3q+6p^2q(2q-1)F$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\times aa$	$p^2q^2+pq(1-6pq)F$	0	1	0
$Aa\times AA$	$2p^3q+6p^2q(2q-1)F$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\times Aa$	$4p^2q^2+4pq(1-6pq)F$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\times aa$	$2pq^3+6pq^2(1-2q)F$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
$aa\times AA$	$p^2q^2+pq(1-6pq)F$	0	1	0
$\times Aa$	$2pq^3+6pq^2(1-2q)F$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
$\times aa$	q^4+6pq^3F	0	0	1

Note that the mating frequencies in Table 1 are only approximations that were obtained by Yasuda [cited in Morton (1963)] by carrying out inbreeding on an inbred population with the same extent of inbreeding (F is constant). To fix our ideas, let the first parent in Table 1 (first column) be the male, followed by the female parent. The joint distribution of parent-child pairs is derived from Table 1, the results of which are summarised in Table 2 [similar to Table 2 of Li (1968)]. For example, we shall derive, the probability of AA-AA cell (father, say, AA and a child AA). This probability is obtained from Table 1 as the sum of products of columns (2) and (3) over the first three matings where the male parent is AA, which is given by

$$p^4+6p^3qF+\tfrac{1}{2}[2p^3q+6p^2q(2q-1)F]$$
$$=p^4+p^3q+3p^2qF(2p+2q-1)$$
$$=p^3+3p^2qF$$

Similarly the other entries of Table 2 are obtained.

TABLE 2

Joint distribution of parent-child pairs in a population under general mating system

			Child			
			AA	Aa	aa	Marginal totals
	$Y \rightarrow$		Y_2	Y_1	Y_0	
Parent $\downarrow$		$Z \rightarrow$	2	1	0	
AA	Y_2	2	p^3+3p^2qF	$p^2q+pq(1-3p)F$	0	p^2+Fpq
Aa	Y_1	1	$p^2q+pq(1-3p)F$	$pq(1-F)$	$pq^2+pq(1-3q)F$	$2pq(1-F)$
aa	Y_0	0	0	$pq^2+pq(1-3q)F$	q^3+3pq^2F	q^2+Fpq
Marginal totals			p^2+Fpq	$2pq(1-F)$	q^2+Fpq	1

(5) From Table 2 we now proceed to obtain the correlation between parent's Z and child's Z which is given by

$$\gamma_{ZZ'_*}=\frac{1+3F}{2(1+F)}=\frac{1}{2}+\frac{F}{1+F}$$

It may be noted that Chakraborty (1969) recently obtained a series of such interesting correlations like correlation between parent's Z and s (number of) childrens' total Z, for any integer s, by the method of moment generating functions.

(6) Again from Table 2, the covariance between parent's Y and child's Y is seen to be

$$\text{Cov}_*(Y, Y')=\tfrac{1}{2}\sigma_L^2+Fpq[3m-Y_1^2+2Y_1Y_2(1-3p)$$
$$+2Y_0Y_1(1-3q)]$$
$$-2FpqSE(Y)-(FpqS)^2$$

where,

$$m=pY_2^2+qY_0^2$$
$$S=(Y_2-Y_1)+(Y_0-Y_1)$$

and $$E(Y)=p^2Y_2+2pqY_1+q^2Y_0$$

which gives the parent-offspring correlation for metric characters by

$$\gamma_{YY_*'} = \text{Cov}_* (Y, Y')/\sigma_{Y_*}^2.$$

Now observe that for $F=0$ (random mating) we get back the results given by Li.

Correlation Between Half-Sibs Under General Mating System

Here, we shall obtain a general expression for the correlation between two half-sibs (γ_{XX_*}) with respect to a phenotypic measurement X. In fact, X denotes the actual (phenotypic) measurements directly observed on the individuals. This case arises when we introduce environmental variation in Y. Denoting the effect of environments on the individual measurements by E, we have

$$X = Y + E \quad \text{and} \quad \sigma_{X_*}^2 = \sigma_{Y_*}^2 + \sigma_E^2$$

Now consider one male parent (husband) and two female parents (wives), and one child from each couple (two half-sibs). Let m and λ denote γ_{ZZ_*} between husband-wife (same for both couples) and the two wives respectively.

In the diagram,

$$C_1 = \sigma^2_{Y_*}/\sigma^2_{X_*} \; ; \; \sqrt{C_1}$$

is a con tant which corresponds to the path coefficient from X to Y (becomes unity if there is no environmental variation in Y). Similarly, $\sqrt{C_2}$ is another constant associated with the path from Y to Z. From the diagram one has

$$\gamma_{XX_*} = \left[\frac{1+3F}{2(1+F)}\right]^2 C_1 C_2 [(1+2m)(1+\lambda)+m^2]$$

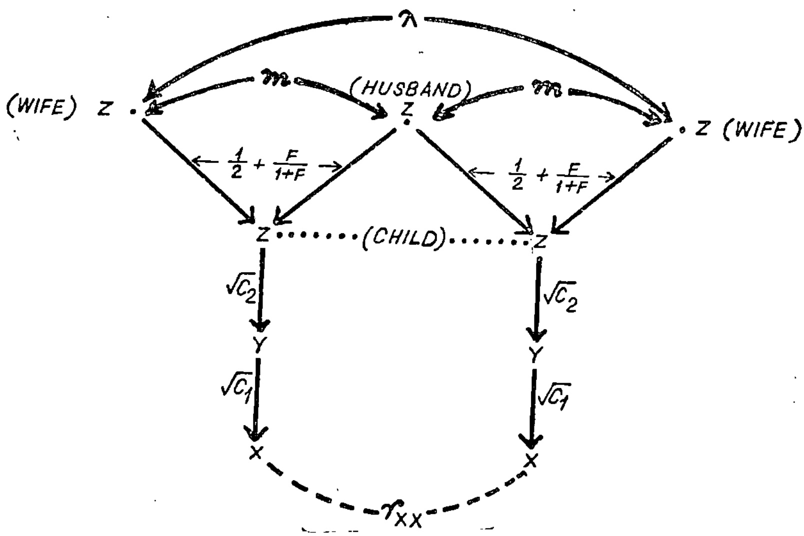
Also if the population is in equilibrium under the general mating system,

$$m = \frac{2F}{1+F}$$

so that we get

$$\gamma_{XX_*} = C_1 C_2 \left[\frac{1+3F}{2(1+F)}\right]^2 \left[(1+\lambda)\left(\frac{1+5F}{1+F}\right) + \left(\frac{2F}{1+F}\right)^2\right]$$

$$= \left[\frac{1+3F}{2(1+F)^2}\right]^2 C_1 C_2 (1+8F+19F^2) \text{ if } m=\lambda$$



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